Quantum criticality of the Lipkin-Meshkov-Glick model in terms of fidelity susceptibility

Ho-Man Kwok (郭灝民),[1] Wen-Qiang Ning (宁文强),[1,2] Shi-Jian Gu (顾世建),[1,*] and Hai-Qing Lin (林海青)[1,2]
[1]*Department of Physics and ITP, The Chinese University of Hong Kong, Hong Kong, China*
[2]*Department of Physics, Fudan University, Shanghai 200433, China*

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We study the critical properties of the Lipkin-Meshkov-Glick model in terms of the fidelity susceptibility. By using the Holstein-Primakoff transformation, we obtain explicitly the critical exponent of the fidelity susceptibility around the second-order quantum phase transition point. Our results provide a rare analytic case for the fidelity susceptibility in describing the universality class in quantum critical behavior. The different critical exponents in two phases are nontrivial results, indicating that the fidelity susceptibility is not always extensive.

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The Lipkin-Meshkov-Glick (LMG) model [1] was introduced in nuclear physics. It describes a cluster of mutually interacting spins in a transverse magnetic field. In condensed matter physics, this model is associated with a system of infinite coordination number. In earlier time, scaling behaviors of critical observables have been studied by mean-field analysis [2], while recently the finite-size scaling of this model was studied by the $1/N$ expansion in the Holstein-Primakoff single boson representation [3] and by the continuous unitary transformations (CUT) [4–6]. Meanwhile, a rich structure of four different regions is revealed in the parameter space through a careful scrutiny on the spectrum [7]. Besides, the quantum criticality has been investigated by studying its entanglement properties [8–12]. Both the first- and second-order quantum phase transitions (QPTs) [13] have been revealed, in the antiferromagnetic and the ferromagnetic cases, respectively [8,9].

Regarding the QPT itself, the ground state of a system would undergo a significant structural change at a certain critical point. This primary observation suggests a new description of QPTs in terms of fidelity [14–26], a concept introduced in quantum information theory [12]. Mathematically it is the overlap between two ground states in which their driving parameters deviated by a small amount. However, the fidelity depends computationally on an arbitrarily small yet finite change of the driving parameter. For this, Zanardi *et al.* introduced the Riemannian metric tensor [18], while You *et al.* suggested the fidelity susceptibility [19], both focus on the leading term of the fidelity, in order to explain singularities in QPTs. In addition, scaling analysis of these quantities has been informative: it helps understanding their divergence and the criticality of the system [21], and it also reveals the intrinsic relation between the critical exponent of some physical quantities and that of the fidelity susceptibility [22].

In this paper, we explicitly compute the ground-state fidelity susceptibility and its critical exponent of the LMG model. Numerical analysis is also performed to check with our analytic calculations. We show that the $1/N$ expansion in the Holstein-Primakoff transformation is sufficient to determine the critical exponent of the fidelity susceptibility χ_F. In addition, we revealed two distinct critical exponents in two phases which is not a general feature of the fidelity susceptibility. Therefore our findings not only suggest another route to understanding the quantum criticality of the LMG model, but also show that the fidelity susceptibility is not always extensive in the critical phenomena of a quantum many-body system.

The Hamiltonian of the LMG model reads

$$H = -\frac{\lambda}{N}\sum_{i<j}(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j) - h\sum_i \sigma_z^i \tag{1}$$

$$= -(\lambda/N)(1+\gamma)(\mathbf{S}^2 - S_z^2 - N/2) - 2hS_z - (\lambda/2N)(1-\gamma)(S_+^2 + S_-^2), \tag{2}$$

where $\sigma_\kappa(\kappa=x,y,z)$ are the Pauli matrices, $S_\kappa=\Sigma_i\sigma_\kappa^i/2$, and $S_\pm=S_x\pm iS_y$. The prefactor $1/N$ is necessary to ensure finite energy per spin in the thermodynamic limit. It is understood that the total spin and the parity $P=\Pi_i\sigma_z^i$ are the conserved quantities, i.e., $[H,\mathbf{S}^2]=[H,P]=0$. In addition, in the isotropic case $\gamma=1$, one has $[H,S_z]=0$ and simultaneous eigenstates can be found. In the main context, the following parameter space is considered: $\lambda=1$, $|\gamma|<1$, $h\geqslant 0$. We take $h\geqslant 0$ as the spectrum invariant under the transformation $h\leftrightarrow -h$. As a common practice we only consider the maximum spin sector $S=N/2$ which contains the lowest energy state.

We now briefly review of the concept of the fidelity susceptibility here. Suppose there is a Hamiltonian of a general form as

$$H = H_0(\gamma) + f(h)H_I, \tag{3}$$

for H_I is defined as the driving term of the system, which simply does not commute with H_0. The function $f(h)$ coupled to H_I is often considered as the linear external field $f(h)=h$. Then the fidelity susceptibility is defined as [18,19]

$$\chi_F = \left[\frac{df(h)}{dh}\right]^2 \sum_{n\neq 0}\frac{|\langle n|H_I|0\rangle|^2}{[E_n - E_0]^2}, \tag{4}$$

where E_n and $|n\rangle$ stand for the nth eigenenergies and eigenstates of the (whole) Hamiltonian, respectively.

The fidelity susceptibility is well defined for a nondegenerate ground state and non-eigen-driving Hamiltonians H_I. However, the LMG model undergoes ground state level crossing when $\gamma=1$, and $H_I=-2S_z$ commutes with the whole Hamiltonian. So we put our focus on calculating the fidelity

*sjgu@phy.cuhk.edu.hk

susceptibility for an arbitrary isotropy $|\gamma|<1$. One resolution is to use the Bethe-ansatz solution [27,28], which is rather complicated. So we adopt the $1/N$ expansion method which was used extensively by Dusuel and Vidal [4,5], that corresponds to the large N limit.

The $1/N$ expansion method is done under the Holstein-Primakoff single boson representation [3] framework. In the low energy spectrum the spin operators in the $S=N/2$ subspace are mapped into bosonic operators:

$$S_z = S - a^\dagger a,$$

$$S_+ = (2S - a^\dagger a)^{1/2} a = N^{1/2}(1 - a^\dagger a/N)^{1/2} a = S_-^\dagger, \tag{5}$$

where a ($a^\dagger$) is the standard bosonic annihilation (creation) operator satisfying $[a,a^\dagger]=1$. The above transformation is direct and valid when $h \geqslant 1$; when $0<h<1$ it can also be used through semiclassical treatment [4,5]. This representation is also known as the spin-wave theory. It is well adapted to the computation of the low-energy physics when $\langle a^\dagger a\rangle/N \ll 1$. After inserting these expressions of the spin operators in Eq. (2), one can approximate the square roots as one and express the result in normal ordered form with respect to the boson vacuum state. Keeping terms of order $(1/N)^{-1}$, $(1/N)^{-1/2}$, and $(1/N)^0$ for $h \geqslant 1$ (in which the approximation is justified), the Hamiltonian becomes

$$H = -hN + (2h - 1 + \gamma)a^\dagger a - [(1-\gamma)/2](a^{\dagger 2} + a^2). \tag{6}$$

The above Hamiltonian can be diagonalized by a standard Bogoliubov transformation $a=\sinh(\Theta/2)b^\dagger + \cosh(\Theta/2)b$, where $b(b^\dagger)$ is the quasibosonic annihilation (creation) operator. Taking $\tanh[\Theta(h\geqslant 1)]=(1-\gamma)/(2h-1+\gamma)$, the Hamiltonian becomes

$$H = -h(N+1) + 2\sqrt{(h-1)(h-\gamma)}(b^\dagger b + \tfrac{1}{2}). \tag{7}$$

Thus the low-energy spectrum of the model is mapped to the spectrum of a simple harmonic oscillator. The eigenstates are just $\{|n\rangle\}$, where $b^\dagger b|n\rangle = n|n\rangle$. We consider the driving Hamiltonian H_I responsible for the QPT,

$$H_I = -\sum_i \sigma_z^i = -2S_z. \tag{8}$$

By transforming it into combinations of b and $b^\dagger$ operators, the fidelity susceptibility is calculated as

$$\chi_F = (1-\gamma)^2/[32(h-1)^2(h-\gamma)^2]. \tag{9}$$

The derivation above is only valid for $h\geqslant 1$, for $0<h<1$ the calculation is actually similar to the above case of $h\geqslant 1$, provided that one first rotates the z axis to bring it along the classical spin direction. We do not show it explicitly here, but interested readers are recommended to refer to Refs. [4,5]. We simply quote the main result; after all the procedures the Hamiltonian becomes

$$H = -\frac{(1+h^2)}{2}N - \frac{1-\gamma}{2} + 2\sqrt{(1-h^2)(1-\gamma)}\left(b^\dagger b + \frac{1}{2}\right). \tag{10}$$

The driving Hamiltonian also takes a different form:

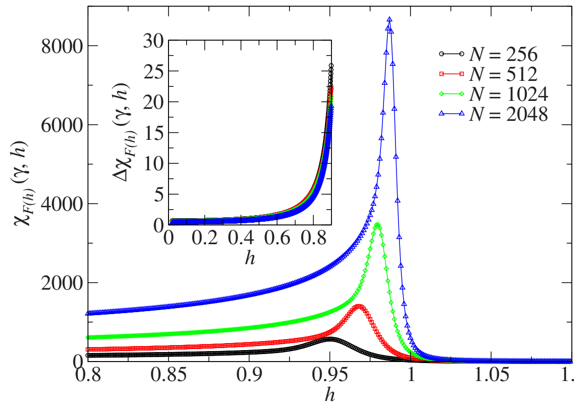

FIG. 1. (Color online) Fidelity susceptibility as a function of h at $\gamma=0.5$. The inset denotes the difference between the fidelity susceptibility and the extensive term in Eq. (12).

$$-\sum_i \sigma_z^i = -2S_z = -2(-\sqrt{1-h^2}\,\widetilde{S_x} + h\widetilde{S_z}), \tag{11}$$

for the Holstein-Primakoff (HP) transformation is done on the $\tilde{S}$ operators. The fidelity susceptibility is then obtained accordingly:

$$\chi_F = \frac{N}{4\sqrt{(1-h^2)(1-\gamma)}} + \frac{h^2(h^2-\gamma)^2}{32(1-\gamma)^2(1-h^2)^2}. \tag{12}$$

Thus we obtained χ_F of the anisotropic LMG model in large N limit. We first see the effect of isotropy to the fidelity susceptibility. It dominates when $h<1$, but fades out for large h. Especially in the isotropic limit, when $\gamma\to 1$, χ_F diverges when $h<1$, but tends to zero when $h>1$. This is the effect of the level-crossing points in the thermodynamic limit. They together form a region of criticality, and the system undergoes continuous level crossing. The fidelity susceptibility responds drastically while moving along h. But when $h>1$, there are no further critical points, χ_F naturally measures zero when moving along h because we have $[H_0,H_I]=0$.

An interesting observation is that χ_F behaves extensively when $h<1$ even in the large N limit. When discarding the extensive part of Eq. (12), we arrive a zero point at $h=\sqrt{\gamma}$, which does not fit with numerical analysis (Fig. 1). This discrepancy may be eliminated by adopting other transformations of the driving Hamiltonian. Particularly, the flow of operators in the LMG model has been studied by the continuous unitary transformation (CUT) method [4,5]. However, such a discrepancy would not hinder us from getting the correct critical exponent of the fidelity susceptibility.

Let us emphasize the intensive property of the fidelity susceptibility, which measures the average response to some driving Hamiltonians. Its divergence should correspond to a critical point of a second-order QPT rather than to the increasing system size. In order to predict the critical exponent correctly, we should average the fidelity susceptibility whenever necessary. To the leading order, Eq. (12) becomes

$$\chi_F/N = 1/\left[4\sqrt{(1-h^2)(1-\gamma)}\right]. \tag{13}$$

Then it comes to a key result of our paper: χ_F bears different critical exponents across the critical point. It diverges as $(1-h)^{1/2}$ when $h<1$, and the critical exponent is $1/2$; $(h-1)^2$ when $h>1$, so that the critical exponent is 2. It is unlike the Ising model in a transverse field [15] and the one-dimensional asymmetric Hubbard model [22], where the critical exponent is a single number over the phases.

To illustrate the scaling behavior of the fidelity susceptibility, we perform the exact diagonalization (ED) to solve the spectrum of H and then calculate the corresponding fidelity susceptibility numerically. Let us recall the fidelity susceptibility scaling analysis performed in the asymmetric Hubbard model [22]. According to the scaling ansatz [29] and the obvious power-law divergence observed in Fig. 1, the rescaled fidelity susceptibility around its maximum point at $h_{\max}$ is a simple function of a scaling variable, i.e.,

$$(\chi_{F\,\max}-\chi_F)/\chi_F = f[N^{\nu}(h-h_{\max})], \tag{14}$$

where $f(x)$ is the scaling function and ν is the correlation length critical exponent. This function is universal and does not depend on the system size, as shown in Fig. 2 for cases of $\gamma=0.5$, 0, and -0.5. Remarkably, the critical exponent ν for the three cases are very close. This observation strongly implies that ν is a universal constant and does not depend on the parameters γ and h.

In recent studies of the fidelity susceptibility in critical phenomena, it was pointed out that the intensive fidelity susceptibility scales generally like [21,22]

$$\chi_F \propto 1/|h-h_c|^{\alpha} \tag{15}$$

around the critical point. In the last section, we have already obtained

$$\alpha = \begin{cases} 2, & h>1 \\ \frac{1}{2}, & 0 \leqslant h < 1, \end{cases} \tag{16}$$

which is also a universal constant. Then if the maximum point of the intensive fidelity susceptibility scales like

$$\chi_{F\,\max} \propto N^{\mu}, \tag{17}$$

the scaling ansatz also implies another important relation, i.e.,

$$\alpha = \mu/\nu. \tag{18}$$

We try to confirm this equality numerically. In Fig. 2, Eq. (14) is best fitted with $\nu \simeq 0.665$. The case to determine μ is more subtle. It is because Eq. (14) remains the same form even for averaged χ_F, but the maximum of χ_F does not. To resolve this problem, we first determine μ from the "bare" χ_F. By using the least square fit method, we evaluated bare μ for different γ. The numerical details are shown in Table I. However, the exponent μ does not converge perfectly. We

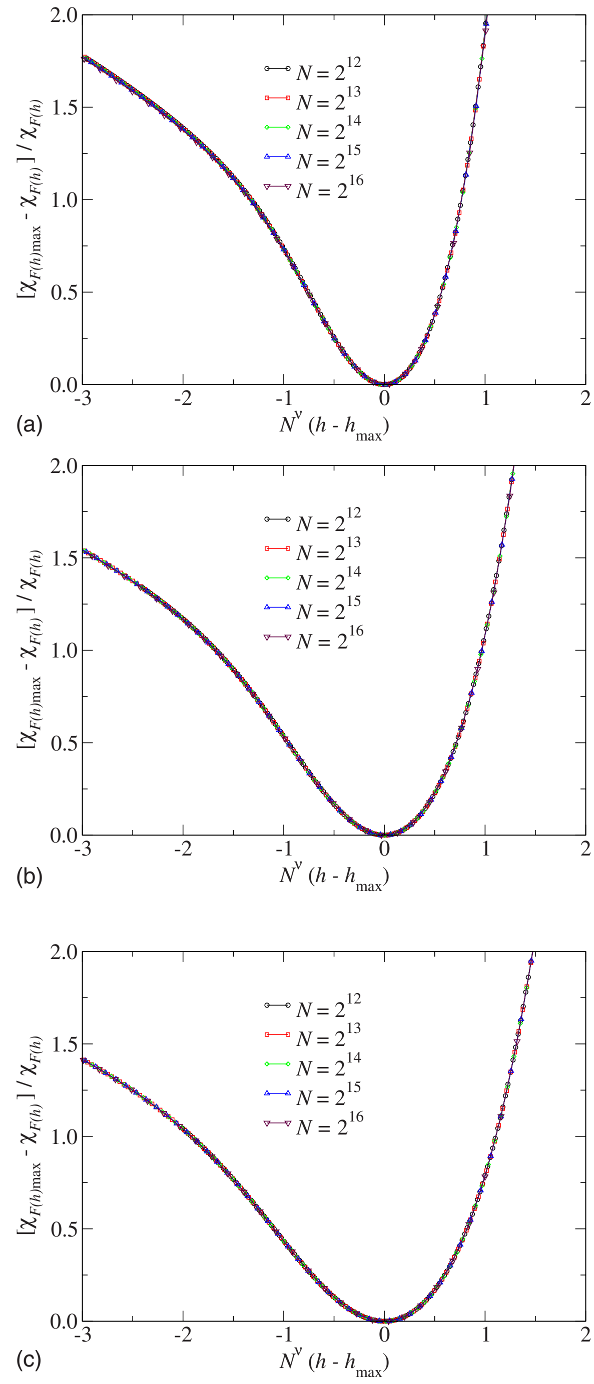

FIG. 2. (Color online) The finite-size scaling analysis is performed for the case of power-law divergence at $\gamma=0.5$ (a), $\gamma=0$ (b), and $\gamma=-0.5$ (c) for system sizes $N=2^n$ $(n=12,13,14,15,16)$. The fidelity susceptibility is considered as a function of the scaling variable $N^{\nu}(h-h_{\max})$, with the correlation length critical exponent $\nu \simeq 0.665$.

TABLE I. Scaling exponent μ at various γ, obtained by sampling system size in different range.

γ	0.8	0.5	0.2	0	−0.2	−0.5
$\mu(N\in[2^8,2^{16}])$	1.3221 ± 0.0006	1.3264 ± 0.0004	1.3267 ± 0.0004	1.3280 ± 0.0004	1.3283 ± 0.0003	1.3285 ± 0.0003
$\mu(N\in[2^{12},2^{16}])$	1.3250 ± 0.0003	1.3285 ± 0.0004	1.3295 ± 0.0002	1.3299 ± 0.0002	1.3302 ± 0.0001	1.3304 ± 0.0001

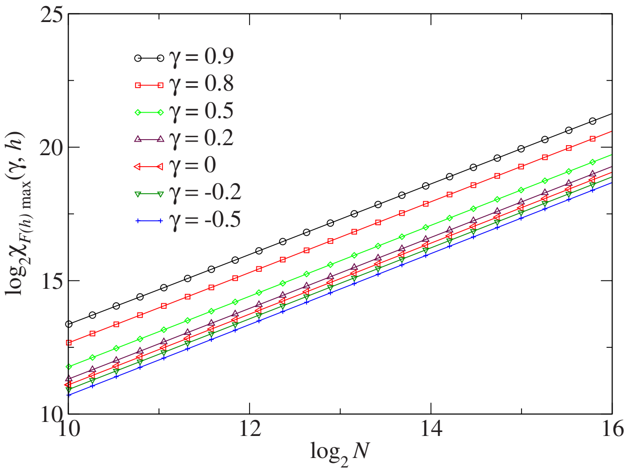

FIG. 3. (Color online) The finite-size scaling is performed for the maximum of the fidelity susceptibility.

compare the μ obtained in a range of $[2^{12}, 2^{16}]$, and those from the range $[2^8, 2^{16}]$. The results converge better for larger scaling regions. According to the trend of μ in larger system sizes, we roughly estimate $\mu = 1.33$ with three effective digits (Fig. 3).

When $h > 1$, χ_F is observed to be intensive (Fig. 1). With the estimated μ and ν, the equality (18) is satisfied with $\alpha = 2$. On the other hand, when $h < 1$, χ_F/N is the intensive quantity. For $\chi_F \propto N^{\mu}$,

$$\chi_F/N \propto N^{(\mu-1)}. \tag{19}$$

Thus $\mu \simeq 0.33$, this will give the relation $\alpha = 1/2$. These two values of α are consistent with our analytic calculation in the last section.

The exponent μ, ν can also be discussed from the scaling ansatz at the critical point rather than the maximum point of a finite system, as shown by Vidal, Dusuel, and Barthel [5,6]. Based on their approach, the critical exponent ν takes the value of $2/3$, and is independent of the magnitude of γ. Our results on the maximum study simply agree with this value and can be generalized to other models where the precise critical point is not known.

Another scaling analysis is to examine how $h_{\max}$ tends to the critical point $h_c = 1$. It should scale like $h_c - h_{\max} \propto N^{-\delta}$ in the large N limit. We find $\delta \simeq 0.66$ with two effective digits for various γ.

In short, we can confirm that the exponents μ, ν, and δ of the fidelity susceptibility do not depend on the value of γ and h. They are universal constants for the LMG model and are related to the critical exponent of the fidelity susceptibility α.

In summary, we computed explicitly the fidelity susceptibility and its critical exponent of the LMG model at different isotropy. We found that the fidelity susceptibility is not always an extensive quantity, indicated by the different critical exponents of the fidelity susceptibility in two phases. Such a rather nontrivial result is further confirmed by ED and the related numerical analysis. Since the fidelity susceptibility is believed to be able to characterize the universality class of quantum phenomena, our results therefore provide a rare explicit case for the study of fidelity susceptibility.

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